Chapt2：Configuration of the Atom: Rutherford’s Model

2.1 Background

2.2 Emergence of the Rutherford’sModel

2.3 Rutherford scattering Formula

2.4 Experimental test of the RF Formula.

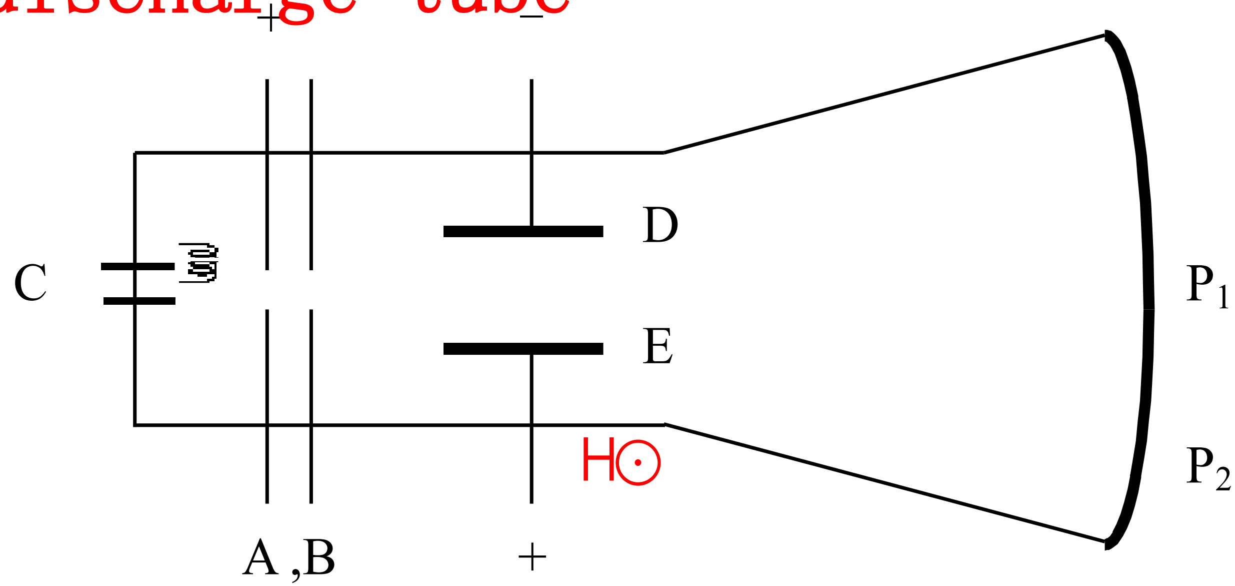
2.5 Summary of the Nuclear planet model

2-1 Background

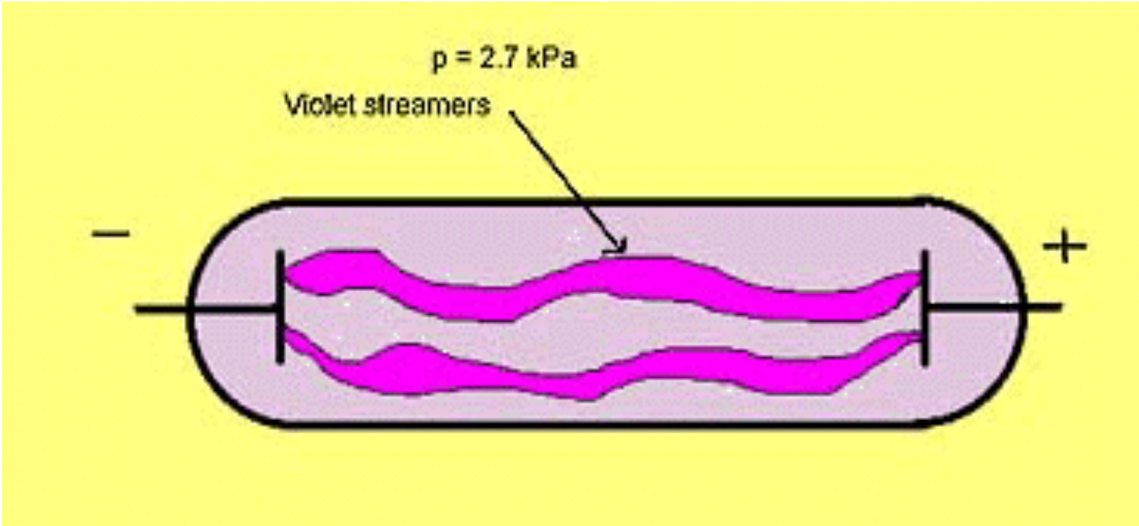
1. **The discovery of the electron**
2. **The charge and mass of the electron**
3. **The size of Atom**

2.1.1 Discovery of the electron

- Thomson's discharge tube



- Cathode rays(C)⇒Slits(A,B)⇒paralle plates(D,E)⇒screen
- D,E with E ⇒ rays P_1 → P_2 ⇒ negative charged
- With H ⇒ rays P_2 → P_1 ⇒ $Hev=Ee$ ⇒ $v=E/H$
- Without E ⇒ rays radius r ⇒ $mv^2/r= Hev$ ⇒ $e/m=v/Hr$



Thomson's Gas discharge tube

- In the late 1800s several experiments were performed on electrical discharges in gases at low pressure.
- It was established that a stream of some kind of "rays" – called cathode-rays – was emitted by any electrode at high negative potential in a vacuum tube.
  - What were these rays?

2.1.2 Charge & Mass of Electron

- Oil-drop experiment by Millikan：
  - Measured charge ： e = 1.6×10^{-19} C
    ⇒electron's mass m_e = 9.1×10^{-31} kg
  - Millikan found charge is quantized
    - **Any charge can only be an integral multiple of *e (smalled charge)***

given $\frac{e}{m_p}$ ⇒ $\frac{m_e}{m_p} = \frac{1}{1836}$

- Oil drop experiment **(2)**
  - Atom is neutral, with negative charged electrons, ⇒ there must be postive charged matter(proton)

Atom = Postive + Negative + Neutral

2.1.3 Avogadro's Constant

- N_A gives the number of atoms in one Mole
- Atomic mass unit [u]
  - 1[u] ≡ The mass of one ^{12}C /12

$$= \frac{12克}{12 \times N_A} = \frac{1}{N_A}[克] \doteq 1.66 \times 10^{-24}[克]$$

  - Atomic mass M_A [u] = atomic quantity [u] = Mass number A [u]

  - 1g= u N_A ; F=e N_A R= k N_A

  - Macro -- N_A --- Micro

2.1.4 The size of atoms

- Atomic radius
  - Atomic volum = $\frac{4}{3}\pi r^3$

= Mass of an atom/atomic density = $\frac{A/N_A}{\rho}$

$\Rightarrow$ radius r = $(\frac{3A}{4\pi\rho N_A})^{1/3}$

order of magnitude: $r \sim 10^{-10}$ m = 1 Å

2.2 Emergence of Rutherford's Model

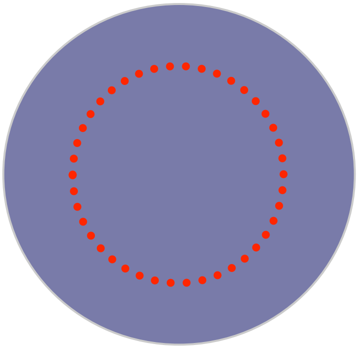
1. Atomic charges' distribution?
2. α scattering
3. Scattering experiments

2.2.1 How are charges distributed in atoms

- Thomas atomic model

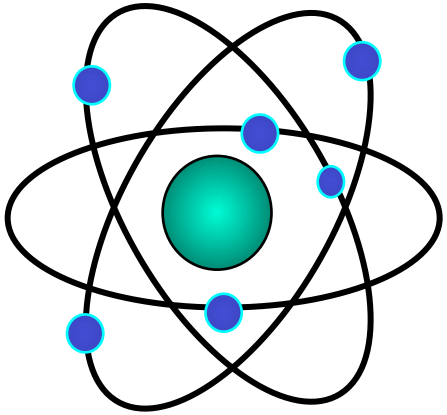
  postive charges are distributed uniformly over the whole atom,

electrons are embedded



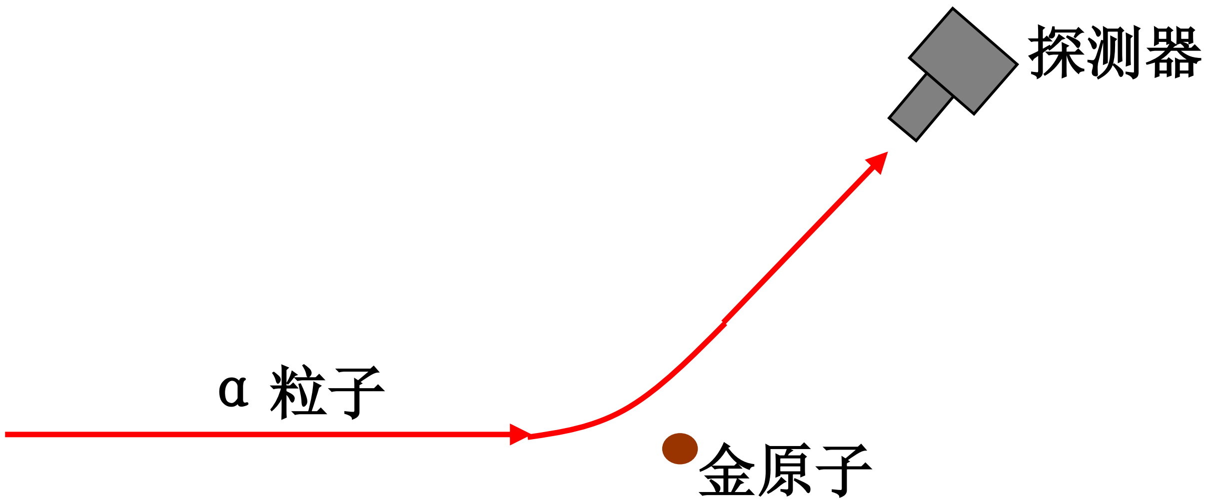
- **Rutherford model**

  Atom is composed of by nuclei and circling around electrons



2.2.2 α scatting experiment

α 粒子散射实验：

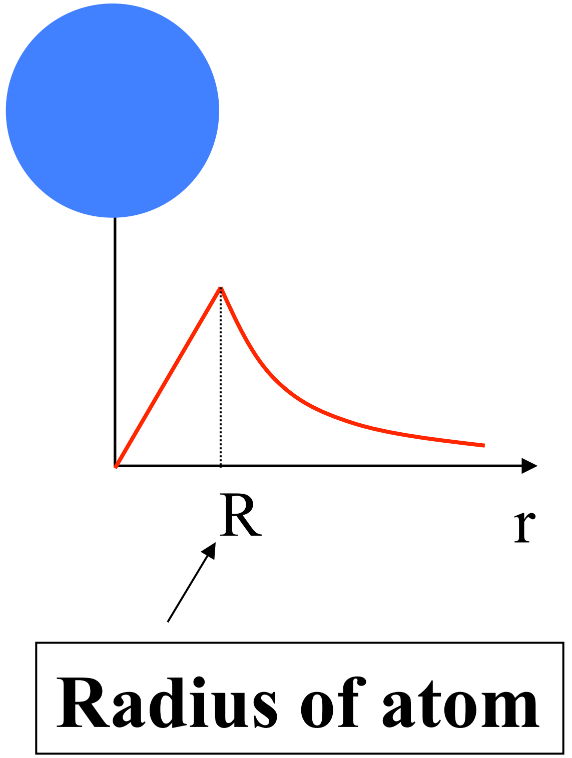


- results:
  - Most *α particles have scattering angle*: ~ 2° - 3°
  - few of 1/8000 α with SA: > 90°
- **puzzled : like a gun bullet was bounced back by a paper**

2.2.3 Quantitative explaination

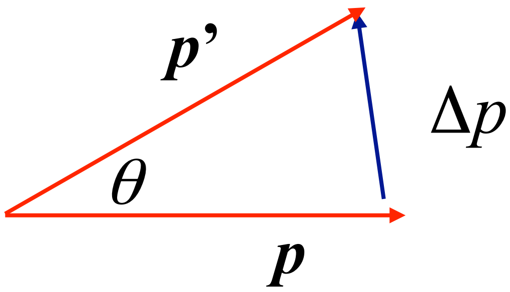
- Thomson's model
  - **Force of Positive charge *Ze of atom* on the** α particle

$$F = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Ze^2}{r^2} & r \geq R \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Ze^2}{R^3}r & r < R \end{cases}$$



The Max. Force of Positive charge *Ze of atom* on the α particle

$$F = \frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{R^2}$$



- Scat. Angle

$$\theta = \frac{\Delta p}{p}$$

– **Momentum change~** F t (2***R/v)***

- So we have

$$\frac{\Delta p}{p} = \frac{2FR/v}{m_\alpha v} = \frac{2Ze^2/(4\pi\varepsilon_0 R)}{\frac{1}{2}m_\alpha v^2}$$

$\boxed{E_\alpha}$, $\boxed{R}$, $\boxed{\dfrac{e^2}{4\pi\varepsilon_0}}$

$$\approx \frac{2Z\times 1.44\text{fmMeV}/0.1\text{nm}}{E_\alpha(\text{MeV})} \approx 3\times 10^{-5}\frac{Z}{E_\alpha}\text{rad}$$

2.2.3 Electron's effect

- Electron and α particle' s (head on)

P and E conservation →

$$m_\alpha v_0 = m_\alpha v_1 + m_e v_e, \quad \frac{1}{2}m_\alpha v_0^2 = \frac{1}{2}m_\alpha v_1^2 + \frac{1}{2}m_e v_e^2$$

In coming α ($m_\alpha v_0$)

Outgoing α ($m_\alpha v_1$)

Outgoing electron ($m_e v_e$)

$$\rightarrow v_e = \frac{2m_\alpha v_0}{m_\alpha + m_e} \approx \frac{2m_\alpha v_0}{m_\alpha} = 2v_0$$

$$\rightarrow \Delta p = m_\alpha v_0 - m_\alpha v_1 = m_e v_e \approx 2m_e v_0$$

therefore

- Momentum change

$$\Delta p = m_\alpha v_0 - m_\alpha v_1 = m_e v_e \approx 2 m_e v_0$$

$$p = m_\alpha v_0$$

$$\rightarrow \frac{\Delta p}{p} \approx \frac{2 m_e v_0}{m_\alpha v_0} = \frac{2 m_e}{m_\alpha} \approx \frac{1}{4000} \approx 10^{-4}$$

Deflection angle of α is very tiny
Almost no contribution

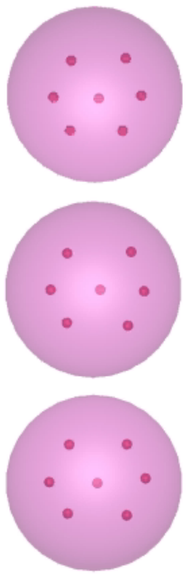
- α−particle on Gold atom

$$E_\alpha = 5\text{MeV} \quad \boldsymbol{Z=79}$$

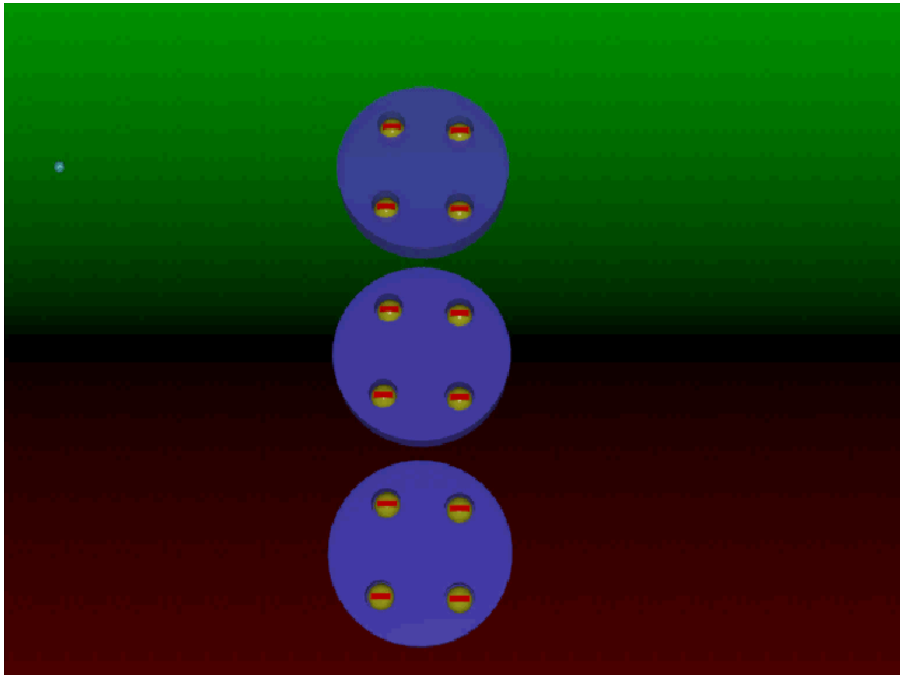
$$\theta = \frac{\Delta p}{p} \approx 3\times 10^{-5}\frac{Z}{E_\alpha}\text{rad} < 10^{-4}\frac{Z}{E_\alpha}\text{rad} < 10^{-3}\text{rad}$$

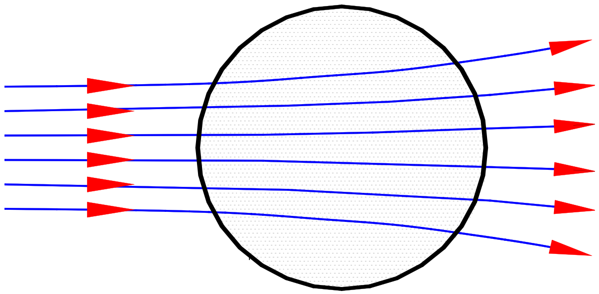
- One scattering angle 10^{-3}rad
- Multiple scatterings do not result large angle
  - Random in direction
- Thomson model contradicts with experiment result !

Thomson model predicts:



- SA 3°probabilty less than 1%
- SA greater than 90°, about 10^{-3500}。





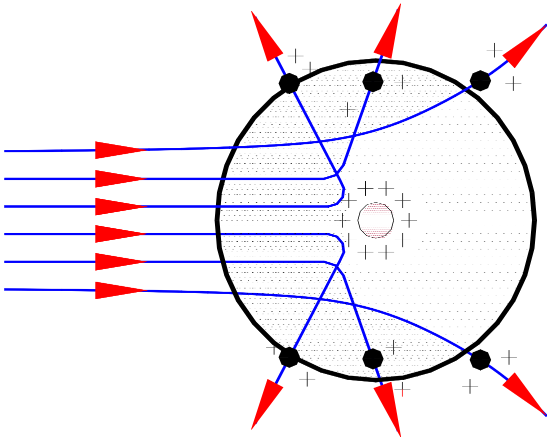
10^{-3500} ---------- 1/8000

R

- most with small SA, about 1/8000 with SA>90°;
- Very few with 180

C

Positive charges locate on the central of atom



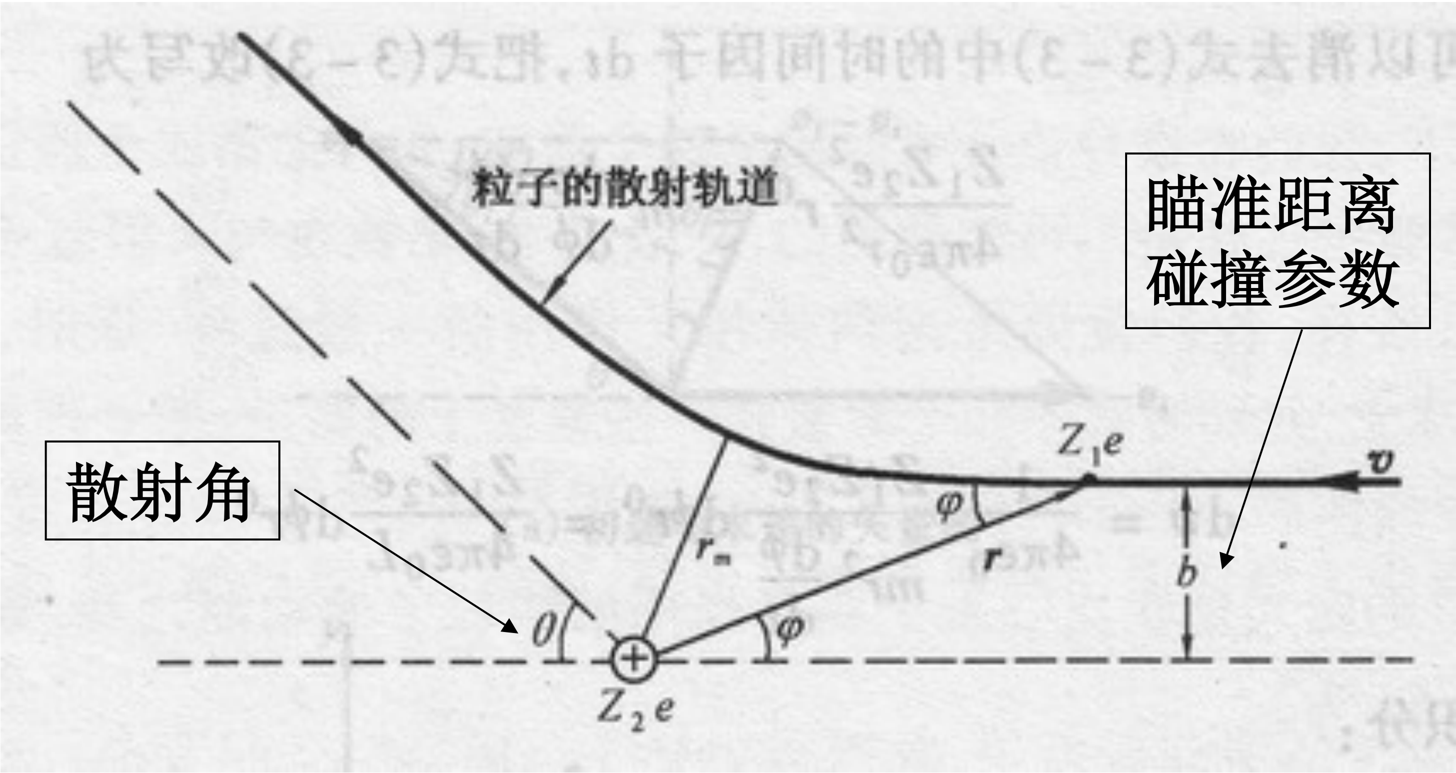
2.3 Rutherford Scattering Formula

1. Derivation of Coulomb formula

2. Derivation of Rutherford formula

2-3-1 Coulomb scattering formula

- **Scattering of particle with E, Z_1e *by a target nucleus* of charge Z_2e**



- Coulomb scattering formula

$$b = \frac{a}{2} ctg \frac{\theta}{2}$$

Impact parameter

$$a = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 E}$$

Scattering factor

- **Assumptions：**
  1. **Single scattering**
  2. **Point charge and only Coulomb force**
  3. **Extranuclear elctrons may be negnected**
  4. **Target nucleus is rest**

- According to

$$\vec{F} = m\vec{a} \to \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 r^2}\vec{r}^{\,0} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

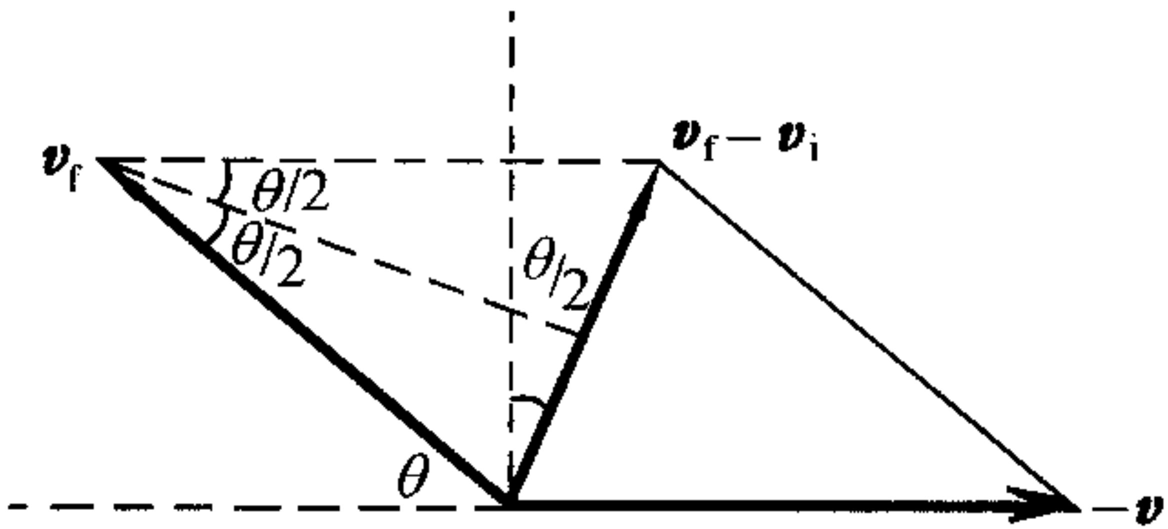
$$mr^2\frac{\mathrm{d}\phi}{\mathrm{d}t} = L \leftarrow$$

Central force→ L conservation

$$\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 r^2}\vec{r}^{\,0} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

$$\mathrm{d}\vec{v} = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{mr^2\frac{\mathrm{d}\phi}{\mathrm{d}t}}\mathrm{d}\phi\,\vec{r}^{\,0} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 L}\mathrm{d}\phi\,\vec{r}^{\,0}$$

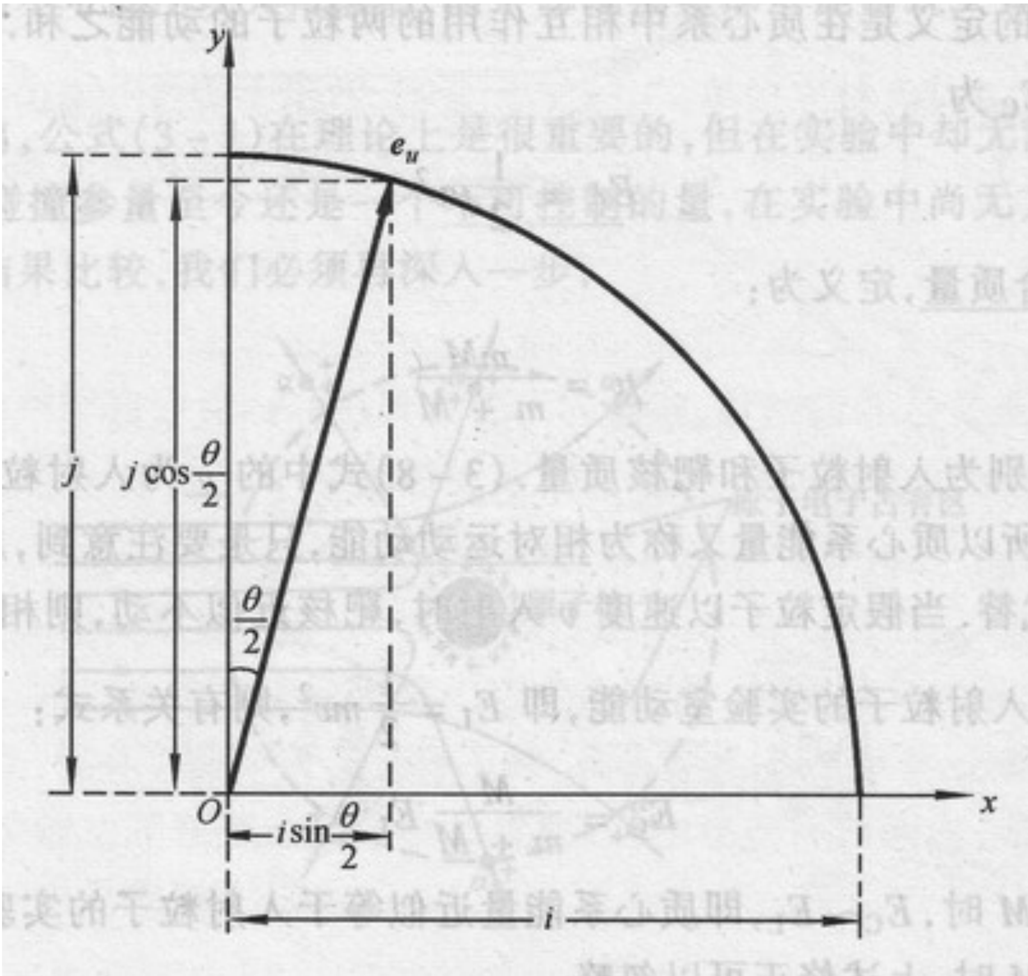
$$\int d\vec{v} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 L}\int \vec{r}^{\,0}\mathrm{d}\phi; \qquad \int \mathrm{d}\vec{v} = \vec{v}_f - \vec{v}_i = \left|\vec{v}_f - \vec{v}_i\right|\vec{e}_u$$



$$E = \frac{1}{2}m\left|\vec{v}_i\right|^2 = \frac{1}{2}m\left|\vec{v}_f\right|^2 \rightarrow \left|\vec{v}_i\right| = \left|\vec{v}_f\right| = v; \quad \left|\vec{v}_f - \vec{v}_i\right| = 2v\sin\frac{\theta}{2}$$

$$\int \vec{r}_0 \, d\phi = \int_0^{\pi-\theta} (\vec{i}\cos\phi + \vec{j}\sin\phi)\mathrm{d}\phi = 2\cos\frac{\theta}{2}\left(\vec{i}\sin\frac{\theta}{2} + \vec{j}\cos\frac{\theta}{2}\right)$$

$\vec{e}_u$



$$v\sin\frac{\theta}{2} = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{L}\cos\frac{\theta}{2} = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{mvb}\cos\frac{\theta}{2}$$

→

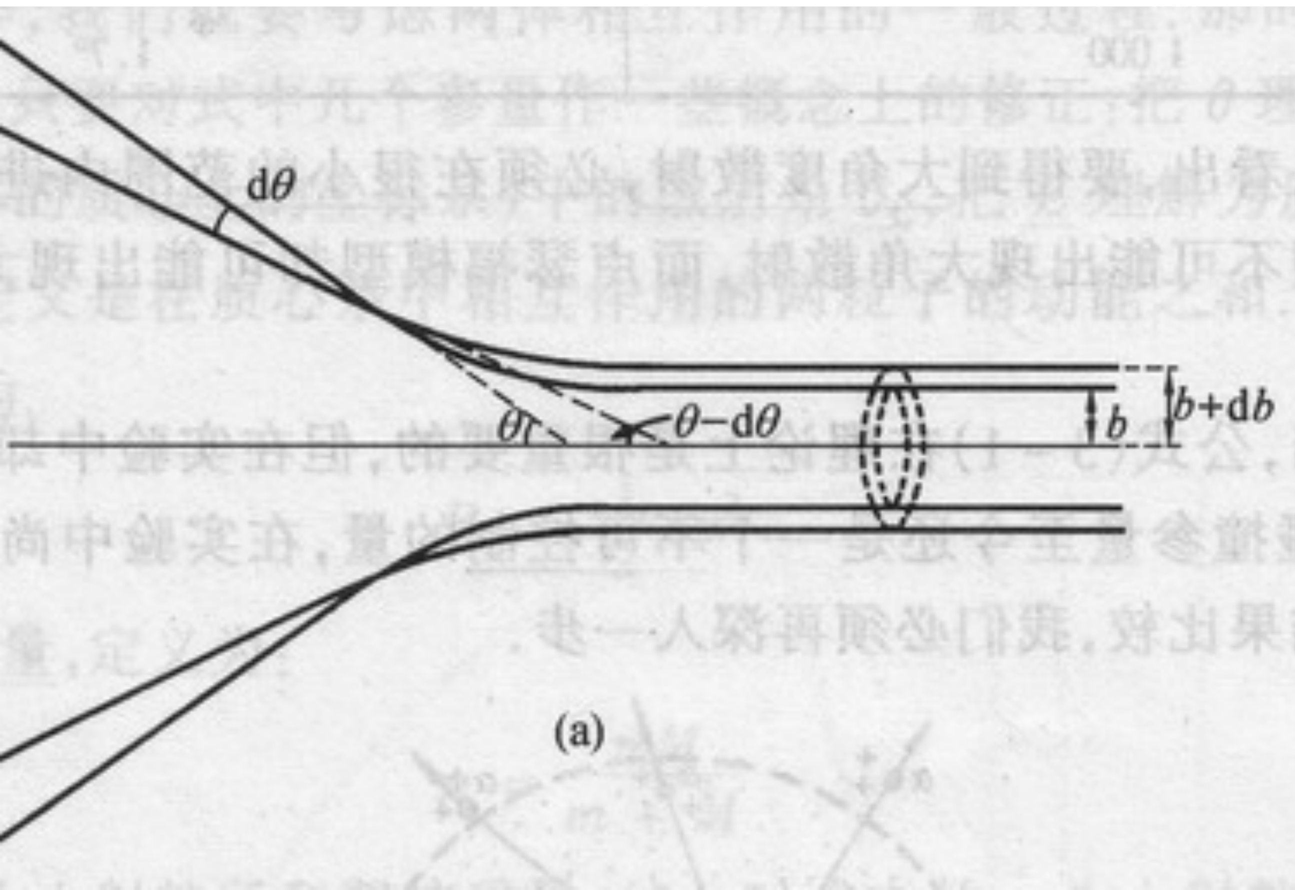
$$\boxed{b = \frac{a}{2}\cot\frac{\theta}{2}} \qquad a = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 E} \qquad E = \frac{mv^2}{2}$$

$b \sim \theta$; b↑→ θ↓; b ↓ → θ ↑

2-3-2 Rutherford's formula (1)

α particle : $b \sim b + db \rightarrow \theta \sim \theta - d\theta$

α: $b \sim b+db$ ring area $\rightarrow \theta \sim \theta\text{-}d\theta$ hollow cone



(a)

prob. on the ring of

- **Target foil with thickness *t*, *A*, number density *n*, mass density *ρ***
- Area of the ring $2\pi b\,|db|$
- α particle on the ring

$$\frac{2\pi b\,|db|}{A} = \frac{2\pi}{A}\frac{a}{2}\cot\frac{\theta}{2}\left|-\frac{a}{2}\csc^2\frac{\theta}{2}\frac{1}{2}d\theta\right|$$

$$= \frac{2\pi a^2 \sin\theta d\theta}{16A\sin^4\dfrac{\theta}{2}}$$

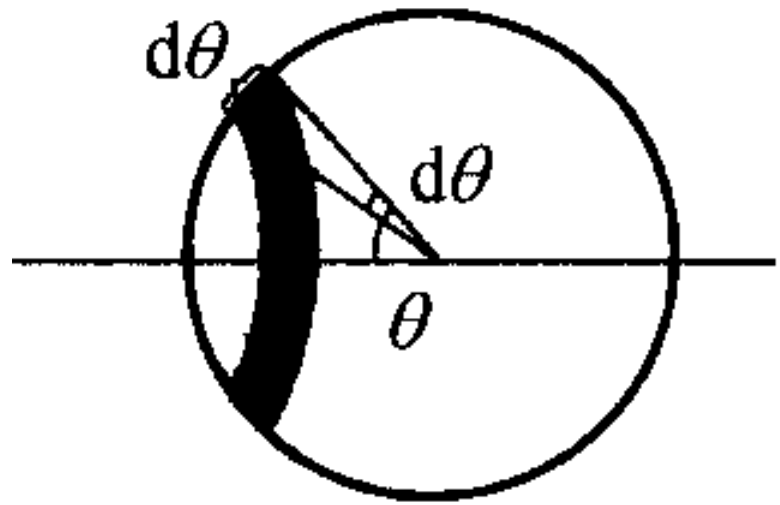
- **Hellow cone cubic angle ~** $d\theta$

$$ds = 2\pi r \sin\theta \; r d\theta;$$

$$d\Omega = \frac{ds}{r^2} = 2\pi \sin\theta d\theta$$

→

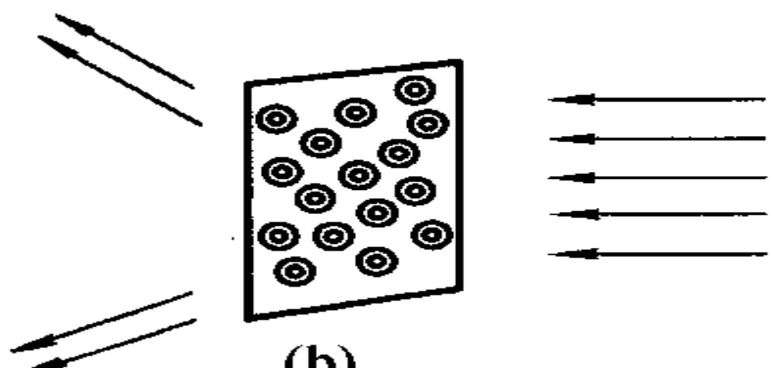
$$\frac{2\pi b\,|\,db\,|}{A} = \frac{a^2 d\Omega}{16 A \sin^4 \frac{\theta}{2}}$$



(c)

- **Many rings in a foil：nucleus ~ring;**
- **volum: *At*; number of rings： *Atn***
- α on ***Atn*** , with the same ***θ***
- The prob. Of one α hiting on the foil with ***θ ~ θ -dθ***

$$dp(\theta) = \frac{a^2 d\Omega}{16 A \sin^4 \frac{\theta}{2}} nAt$$


(b)

- N α scattering on a foil $\theta \sim \theta\text{-}d\theta$

$$dN' = N \frac{a^2 d\Omega}{16A\sin^4\frac{\theta}{2}} nAt = ntN\left(\frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{4E}\right)^2 \frac{d\Omega}{\sin^4\frac{\theta}{2}}$$

- Differential Cross-section

$$\sigma_C(\theta) \equiv \frac{d\sigma(\theta)}{d\Omega} \equiv \frac{dN'}{Ntd\Omega} = \left(\frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

The Ns of scattered particles per number of atoms,per unit area of the target per incident particle per solid angle

- Understanding the D-cross section

$$\sigma_C(\theta) \equiv \frac{d\sigma(\theta)}{d\Omega} \equiv \frac{dN'}{Ntd\Omega} = \left(\frac{1}{4\pi\varepsilon_0}\frac{Z_1Z_2e^2}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

  - **For per unite area of the target nucleus,each incident particle,each unite cubic angle , the number of scatterings**
  - Effective scattering area of every atom into the solid angle element in θ *direction*
  - Dimension : area (m^2/sr)
  - Cross-section: ***b*):** $1b = 10^{-28}m^2, 1mb = 10^{-31}m^2$

2-4 Experimental verification

1. Geiger-Marsden Experiment
2. Nuclear size

Geiger-Marsden Experiment

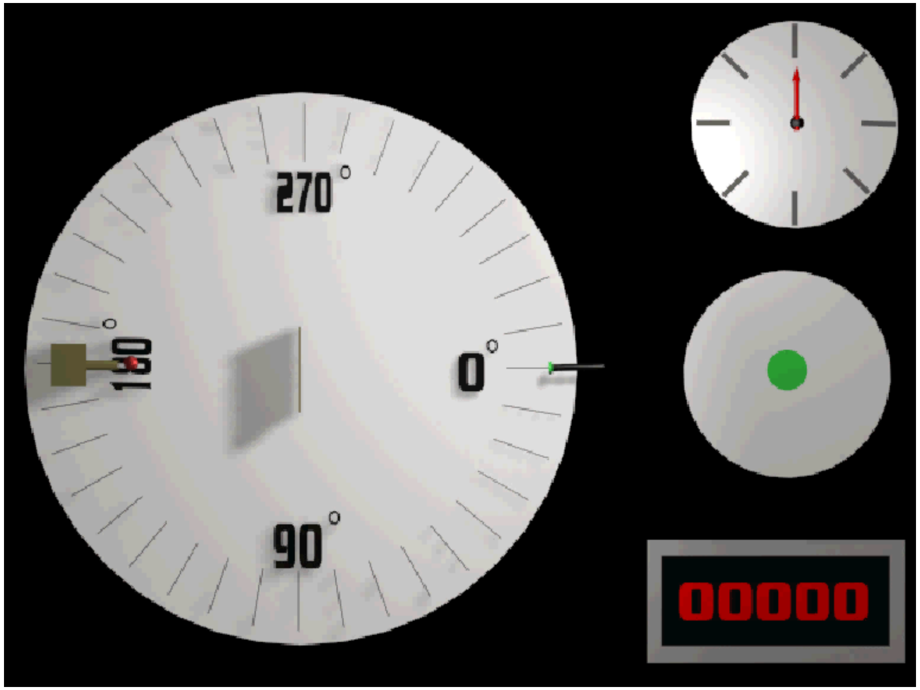
$$\sigma_C(\theta) \equiv \frac{d\sigma(\theta)}{d\Omega} \equiv \frac{dN'}{Ntd\Omega} = \left(\frac{1}{4\pi\varepsilon_0}\frac{Z_1Z_2e^2}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}$$

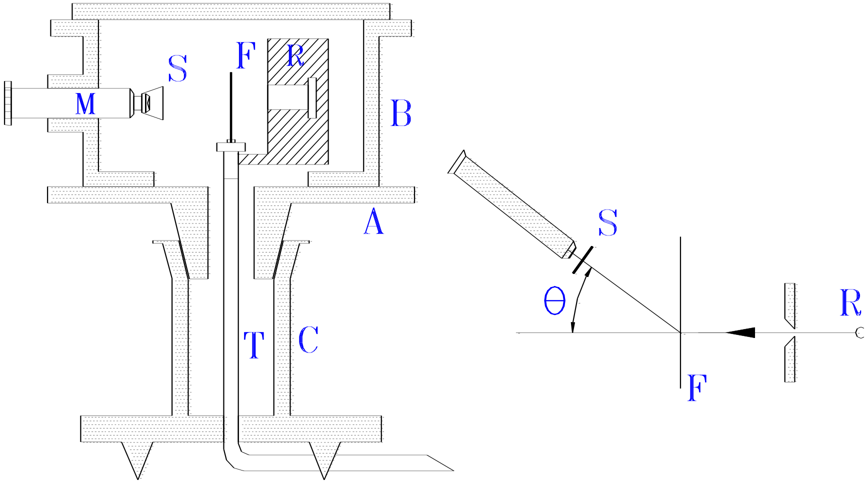
→

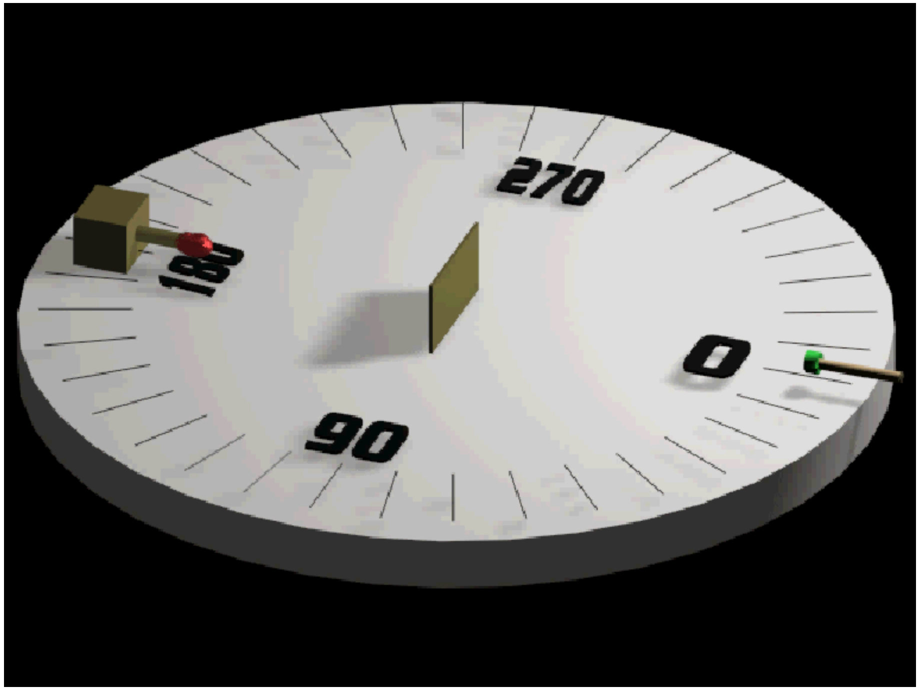
- **fixing** (α source, target) $\frac{dN'}{d\Omega} \propto \frac{1}{\sin^4\frac{\theta}{2}},$
- **Fixing** (α source, material of target, **angle**) $\frac{dN'}{d\Omega} \propto t$
- **same**(target, **angle**) $\frac{dN'}{d\Omega} \propto \frac{1}{E^2}$
- **same**(α source, **angle**, ***nt***) $\frac{dN'}{d\Omega} \sim Z_2^2$

实验装置和模拟实验

- R
- F
- S
- B
- A
- C
- T
- M







- The experiment verified above relations!

2-4-2 Nuclear size estimation (1)

$$\begin{cases} \frac{1}{2}mv_0^2 = E = \frac{1}{2}mv_m^2 + \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 r_m} \\ mv_0 b = mv_m r_m = L \end{cases}$$

$$r_m^2 = \frac{L^2}{2mE} + \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 E} r_m; \quad r_m^2 - a r_m - b^2 = 0$$

$$a = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 E}, \qquad \frac{m^2 v_0^2 b^2}{2m \cdot \frac{1}{2} m v_0^2} = b^2$$

2.4.2 Estimation of the Size(2)

- solve

$$r_m^2 - ar_m - b^2 = 0$$

$$r_m = \frac{a \pm \sqrt{a^2 + 4b^2}}{2}, r_m > 0 \rightarrow$$

$$r_m = \frac{a}{2} + \frac{a}{2}\sqrt{1 + \frac{4b^2}{a^2}} = \frac{a}{2}(1 + \sqrt{1 + \cot^2\frac{\theta}{2}})$$

$$= \frac{a}{2}(1 + \frac{1}{\sin\frac{\theta}{2}})$$

$$b = \frac{a}{2}\cot\frac{\theta}{2}$$

$$\theta = 180^{\circ} \rightarrow r_m = a : 1.20\text{fm}$$

2.5 On Nuclear model

- **Significances:**
  - **Nuclear struncture of atom**
  - **New method ———— black body method**
- difficulties:
  - Stable (accelerating→**radiation**)
  - Identical (atom ←> solar system, IC)
  - Regeneration